Tuza's conjecture for split graphs with two neighborhood types

MathIsEvenEasier

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Abstract

Tuza's conjecture states that the minimum number of edges meeting every triangle of a graph is at most twice the maximum number of edge-disjoint triangles. We prove it for every finite simple split graph whose independent-side vertices of degree at least two have at most two distinct neighborhoods. The clique size, the neighborhood overlap, and both multiplicities are unrestricted. The proof first truncates multiplicities without changing either optimum. Matching decompositions and sums of vertex labels give triangle packings; cuts give triangle covers. We compare these bounds by exact finite checks for clique sizes at most 42 and a rational interval certificate for all larger sizes. The graph reductions, certificate soundness, domain coverage, and final theorem are formalized in Lean.

1 Statement and proof outline

For a finite simple graph G, a *triangle packing* is a family of pairwise edge-disjoint triangles. A *triangle cover* is a set of edges meeting every triangle. Write $\nu_{\triangle}(G)$ for the largest packing size and $\tau_{\triangle}(G)$ for the smallest cover size. Tuza's conjecture, posed in 1981, asserts that $\tau_{\triangle}(G) \le 2\nu_{\triangle}(G)$ for every G.

A *split partition* is a partition $V(G) = C \dot\cup I$ in which C is a clique and I is independent. A vertex of I is *active* if it has at least two neighbors in C; otherwise it belongs to no triangle. Its *type* is its neighborhood in C.

Theorem 1. *Let G be a finite simple graph with a split partition $V(G) = C \dot\cup I$. If the active vertices of I have at most two distinct neighborhoods in C, then*

$$\tau_{\triangle}(G) \le 2\nu_{\triangle}(G).$$

The theorem imposes no bound on $|C|$ or on the number of vertices of either type. It proves a case of Tuza's conjecture; the general conjecture and the case of arbitrary split graphs remain open. Bonamy et al. [1] proved the conjecture for threshold graphs, which include split graphs with nested outside neighborhoods. Chahua and Gutiérrez [2] proved it for split graphs of order n and minimum degree at least $3n/5$. Zeng [3] proved the two-type case for a specified clique part of size eight, allowing arbitrary multiplicities. Theorem 1 removes that bound on the clique size. Huang et al. [4] subsequently reported the case of three active neighborhood types with a specified eight-vertex clique part. The two results relax different restrictions: here the number of types is at most two, while the clique size is arbitrary.

The argument has two computational parts with explicit graph-theoretic inputs. Section 2 bounds the multiplicities while preserving both optima. Sections 3 and 4 construct packings and covers. Section 5 compares their sizes for $2 \le |C| \le 42$. For larger cliques, Section 6 proves several parameter ranges directly; Section 7 covers the remaining range by one certificate on a bounded real domain. That certificate handles every $|C| \ge 43$ at once. Section 8 describes the Lean theorem and its public source build.

Notation. After removing inactive vertices, write $S, T \subseteq C$ for the two neighborhoods and p, q for their multiplicities. Put

$$k = |C|, \quad s = |S|, \quad t = |T|, \quad u = |S \cap T|,$$

and partition the core into

$$A = S \setminus T, \quad B = T \setminus S, \quad U = S \cap T, \quad O = C \setminus (S \cup T).$$

Their sizes are $s - u, t - u, u, k - s - t + u$. Thus, after ordering $s \le t$,

$$2 \le s \le t \le k, \qquad \max(0, s + t - k) \le u \le s.$$

A single type is represented by $T = S$ and $q = 0$; the clique alone by $p = q = 0$. For $k \ge 2$ these conventions allow a dummy set S of size at least two. Write $z_+ = \max(0, z)$, and set

$$c_k = \binom{k}{2} - \lfloor k^2/4 \rfloor, \qquad g(k) = \begin{cases} 0, & k \le 2, \\ \lceil (k-1)(k-2)/6 \rceil, & k \ge 3. \end{cases}$$

Both are bounds used below: c_k is the number of core edges deleted by a balanced cut, and $g(k)$ is a guaranteed clique packing size.

2 Compressing equal neighborhoods

Let H be any finite simple graph and let $S \subseteq V(H)$ have size $s \ge 2$. Form G_m by adding m independent vertices, each with neighborhood exactly S. The set S need not be a clique. Define

$$h(s) = \chi'(K_s) = \begin{cases} s - 1, & s \text{ even}, \\ s, & s \text{ odd}. \end{cases}$$

For completeness, the required round-robin decomposition is explicit. If s is even, label the vertices by $\{\infty\} \cup \mathbb{Z}/(s-1)\mathbb{Z}$. For each residue c, take the matching

$$\{\{\infty, c\}\} \ \cup \ \{\{c + i, c - i\} : 1 \le i \le (s-2)/2\}.$$

It is perfect: the nonzero residues modulo the odd number $s - 1$ occur exactly once in pairs $\{i, -i\}$. Every edge not incident with ∞ belongs to exactly one matching, because its endpoints a, b determine $c = (a + b)/2$ modulo $s - 1$; division by 2 is valid for an odd modulus. Edges incident with ∞ have their evident unique color. This also covers $s = 2$, when the second set is empty. For odd s, use this construction on $s + 1$ vertices and delete one vertex. The resulting s matchings each have $(s - 1)/2$ edges. Thus every color class has $\lfloor s/2 \rfloor$ edges. The degree lower bound for even s, and edge counting with at most $\lfloor s/2 \rfloor$ edges per color for odd s, also prove that the displayed number of colors is minimal.

Theorem 2 (Exact multiplicity compression)**.** *For every $m \ge 0$,*

$$\nu_\triangle(G_m) = \nu_\triangle(G_{\min(m, h(s))}), \qquad \tau_\triangle(G_m) = \tau_\triangle(G_{\min(m, s-1)}).$$

Consequently truncating at $h(s)$ preserves both parameters simultaneously.

Proof. Both parameters are nondecreasing under addition of vertices, so it suffices to prove the reverse inequalities above their proposed thresholds.

Take any triangle packing in G_m. Every triangle using a new vertex uses one edge of $H[S]$. Collect these edges into a graph $J \subseteq H[S]$. They are all distinct. Restrict a fixed

$h(s)$-edge-coloring of K_S to J. Replace the center of each such triangle by the new vertex indexed by the color of its clique edge. Triangles assigned to the same center use a matching, so their spokes are distinct. Triangles assigned to different centers have different spokes, and their clique edges are distinct. All triangles lying inside H are left unchanged; they use none of these clique edges and no new spokes. Thus the packing survives on $h(s)$ new vertices with the same cardinality.

For covers, let $r = s-1$ and take any triangle cover F of G_r. Put $F_H = F \cap E(H)$ and let J consist of the edges of $H[S]$ not in F_H. For every new vertex x, the set $Z_x = \{v \in S : xv \in F\}$ is a vertex cover of J: the triangle formed by x and an edge of J must be met by a selected spoke. Choose x_* minimizing $|Z_x|$. Every edge of J is incident to Z_{x_*}, and each vertex of S has at most $s-1$ neighbors in S. Consequently

$$|E(J)| \le (s-1)|Z_{x_*}| \le \sum_{x=1}^{r} |Z_x|.$$

The sum counts distinct selected spokes. Replace them all by $E(J)$. The resulting set $F_H \cup E(J)$ has size at most $|F|$, still covers every triangle in H, and contains every edge of $H[S]$. It therefore covers the triangles centered at arbitrarily many new vertices. This directly transforms every cover, without introducing a separate vertex-cover optimum. □

Apply the theorem to one type at a time, taking all other vertices as the old graph H. We may henceforth assume

$$0 \le p \le h(s), \qquad 0 \le q \le h(t).$$

The two types need not be disjoint or incomparable for this reduction.

3 Packing bounds

A triangle meeting I has one outside vertex, called its center, and one core edge. Assigning a matching of core edges to a center produces edge-disjoint triangles: the matching condition prevents reuse of a spoke. Assigning disjoint sets of core edges to different centers prevents reuse of a core edge. In particular, $r \le h(d)$ centers with a common d-vertex neighborhood support $r\lfloor d/2 \rfloor$ triangles, by assigning r matching classes of the coloring above.

3.1 Two ways to select matching classes

Put $e_S = \binom{s}{2}$, $e_T = \binom{t}{2}$ and $e_U = \binom{u}{2}$. Independently choose p of the $h(s)$ matching classes on S and q of the $h(t)$ matching classes on T, uniformly among such subsets. An edge in both selections is kept only for the first type. Deleting edges preserves the matching property. Each edge in S, respectively T, is selected with probability $r_S = p/h(s)$, respectively $r_T = q/h(t)$. The expected number of distinct selected edges is $r_S e_S + r_T e_T - r_S r_T e_U$. Some selection therefore gives an outside-centered packing of size at least

$$L = \lceil r_S e_S + r_T e_T - r_S r_T e_U \rceil . \tag{1}$$

The second construction coordinates the colors on the intersection.

Theorem 3 (Two local colorings with controlled overlap)**.** *Let $s, t \ge 2$, $d = \max(s,t)$, and $H = h(d)$. Put*

$$\bar{p} = \min(p, H), \qquad \bar{q} = \min(q, H), \qquad z = \max(0, \bar{p} + \bar{q} - H).$$

There is an outside-centered packing of size at least

$$\boxed{L_{\mathrm{c}} = \left\lceil \frac{\bar{p}\binom{s}{2} + \bar{q}\binom{t}{2} - z\binom{u}{2}}{H} \right\rceil.}$$

If $s = t$ and $p + q \le h(s)$, the maximum size of an outside-centered packing is exactly

$$\boxed{(p+q)\lfloor s/2 \rfloor.}$$

These statements allow arbitrary overlap and arbitrary o. The last identity concerns only outside-centered triangles, not the maximum packing in the whole graph.

Proof. Fix an H-edge-coloring of K_d. Embed each of S, T injectively in its vertex set, using the same labels for vertices of $S \cap T$. This is possible because the labels used by $S \setminus T$ and $T \setminus S$ may overlap: these are two separate embeddings, not an embedding of their union. Pull back the coloring to each clique. Each local color class is a matching, and a shared edge receives the same color in both cliques. There is no requirement that the coloring be proper on $K_{S \cup T}$.

Identify the colors with $\mathbb{Z}/H\mathbb{Z}$. Fix palettes A, B of sizes $\bar{p}, \bar{q}$ with $|A \cap B| = z$: choose B from the complement of A until it is exhausted, then use the necessary colors of A. For each $r \in \mathbb{Z}/H\mathbb{Z}$, use the palettes

$$A_r = \{c : c + r \in A\}, \qquad B_r = \{c : c + r \in B\}.$$

They retain the prescribed sizes and intersection. Assign their colors to distinct outside centers within each type. Retain any edge selected in both types only for the first type. All remaining triangles are edge-disjoint: within a center their core edges form a matching, between centers the spokes are distinct, and no core edge is retained twice.

Let J_r, K_r be the selected core-edge sets. As r runs through all H shifts, each edge of S is selected exactly $\bar{p}$ times and each edge of T exactly $\bar{q}$ times. Each common edge is selected by both exactly z times, because its color agrees in the two neighborhoods. Summing the union identity gives

$$\sum_r |J_r \cup K_r| + z\binom{u}{2} = \bar{p}\binom{s}{2} + \bar{q}\binom{t}{2}.$$

A shift maximizing the union has size at least its finite average. Taking a ceiling proves the bound. This is a deterministic finite-sum argument; no probability formalization is needed.

If $s = t$, every color class in the round-robin factorization of K_s has size $\lfloor s/2 \rfloor$. When $p + q \le H$, assign disjoint color sets to the two types. No shared edge is selected twice, so every outside center attains this many triangles. No outside center can support more: its triangles must use disjoint pairs of its s spokes. This proves exactness. □

Lemma 4 (Greedy completion in the core)**.** *Let G be any split graph with a k-vertex clique and an outside-centered packing of size ℓ. Define*

$$c_k = \binom{k}{2} - \lfloor k^2/4 \rfloor.$$

Then this particular packing can be extended to a packing of size at least

$$\boxed{\max\left\{\ell, \left\lceil \frac{c_k + 2\ell}{3} \right\rceil\right\}.}$$

There is no restriction on neighborhood types.

Proof. Delete the ℓ core edges used by the given packing. Repeatedly choose a triangle in the remaining core and delete its three edges, until there is no triangle. If b triangles were chosen, the leftover graph has $\binom{k}{2} - \ell - 3b$ edges. Mantel's bound gives

$$\binom{k}{2} - \ell - 3b \le \lfloor k^2/4 \rfloor, \qquad \ell + b \ge \left\lceil \frac{c_k + 2\ell}{3} \right\rceil.$$

The new triangles use neither selected core edges nor any spokes, so the two packings combine. The lower bound ℓ is automatic.

For completeness, Mantel's bound here has a short proof. In a triangle-free graph with $k > 0$ vertices and $e > 0$ edges, every edge vw satisfies $d(v) + d(w) \le k$. Summing over edges and applying Cauchy–Schwarz gives

$$4e^2/k \le \sum_v d(v)^2 = \sum_{vw \in E} (d(v) + d(w)) \le ke.$$

Hence $e \le k^2/4$; the zero cases and the integer rounding are immediate. □

3.2 Packing triangles by sums of labels

Lemma 5 (Triangle packing by sums of labels)**.** *Let H be a finite simple graph on $n \ge 1$ vertices, with $t(H)$ triangles. Its triangles can be assigned n colors so that triangles of the same color are edge-disjoint. Consequently,*

$$\nu_\triangle(H) \ge \left\lceil \frac{t(H)}{n} \right\rceil.$$

In particular, applying this to K_n gives the same guaranteed value $g(n) = \lceil (n-1)(n-2)/6 \rceil$ for $n \ge 3$.

Proof. Label the vertices bijectively by $\mathbb{Z}/n\mathbb{Z}$. Color each triangle by the sum of its three labels. If two triangles share an edge with labels a, b and have the same color, their third labels c, d satisfy $a + b + c = a + b + d$, hence $c = d$. The triangles are therefore equal. Each color class is an edge-disjoint packing. All $t(H)$ triangles lie in these n classes, so the largest class has at least $t(H)/n$ triangles; its size is an integer. For K_n, use $t(K_n) = \binom{n}{3}$ and $\binom{n}{3}/n = (n-1)(n-2)/6$. □

For later use define

$$W(r, z) = \left\lceil \frac{\binom{k}{3} + r\binom{s}{2} + z\binom{t}{2}}{k + r + z} \right\rceil.$$

This counts all triangles in the induced graph consisting of the core, r centers of type S, and z centers of type T, then applies Lemma 5. In particular, $W(p, q)$, $W(p, 0)$ and $W(0, q)$ are lower bounds on $\nu_\triangle(G)$. Combining the preceding constructions gives

$$\begin{aligned} L_* &= \max(L, L_c), \\ N &= \max\left\{ g(k), L_*, \left\lceil \frac{c_k + 2L_*}{3} \right\rceil, W(p, q), W(p, 0), W(0, q) \right\}, \\ \nu_\triangle(G) &\ge N. \end{aligned} \tag{2}$$

Each term is the size of an available packing or a lower bound on such a size; the maximum does not require the different constructions to be compatible with one another.

4 Cover bounds

Lemma 6 (An explicit cover from any core cut)**.** *Let C be the k-vertex core, let $S, T \subseteq C$ be the two neighborhoods, and let their multiplicities be $p, q \geq 0$. For every $A \subseteq C$, with $j = |A|$, there is a triangle edge cover F with*

$$|F| = \binom{j}{2} + \binom{k-j}{2} + p \min\{|S \cap A|, |S \setminus A|\} + q \min\{|T \cap A|, |T \setminus A|\}.$$

Consequently the displayed expression is an upper bound on $\tau_\triangle(G)$. No optimality of this cut or of this particular cover is asserted.

Proof. Give the core parts A and $C \setminus A$ different colors. Put each outside type on the side containing fewer of its neighbors. Delete all edges whose endpoints have the same color. Every triangle has a same-color pair, so the deleted edges meet every triangle.

The deleted core edges are the pairs within A and the pairs within $C \setminus A$, counted by the two binomial terms. For an outside vertex of type S, the deleted spokes are exactly its neighbors on its assigned side, giving the smaller of the two intersection counts; the argument for type T is identical. Distinct outer vertices give distinct spokes, and no spoke is a core edge. Thus the displayed sum is the exact number of deleted edges. □

4.1 Balanced cuts

Define

$$f_k(d) = \max\{0, d/2 - k/8, d - k/2\}.$$

Lemma 7 (A separable cover bound for two arbitrary sizes)**.** *For two neighborhoods $S, T \subseteq C$ of any sizes s, t, with multiplicities $p, q \geq 0$,*

$$\boxed{\tau_\triangle(G) \leq c_k + p f_k(s) + q f_k(t).}$$

No restriction on their intersection or on the unused core is needed.

Proof. We first allow a core vertex to be split fractionally between two parts, each of total size $k/2$. Put

$$b_s = f_k(s), \quad b_t = f_k(t), \qquad r_s = s - b_s, \quad r_t = t - b_t, \quad u = |S \cap T|.$$

The definition implies

$$0 \leq b_s \leq s/2, \quad r_s \leq k/2, \qquad 0 \leq b_t \leq t/2, \quad r_t \leq k/2, \qquad 2(b_s + b_t) \geq s + t - k/2.$$

We construct a fractional cut and fix which part contains each outside type, so that the expected spoke deletion costs per center are at most b_s, b_t, respectively.

If $u \leq b_s + b_t$, retain r_s units of S on one side and r_t units of T on the opposite side. Disjoint choices of these retained portions exist: first use exclusive vertices, then the common region. The required common amounts are $(u - b_s)_+$ and $(u - b_t)_+$, whose sum is at most u. Each retained portion has size at most $k/2$, so fill the remaining spaces to obtain two equal parts. Put each outside type opposite its designated retained portion. Its deleted spoke mass is at most b_s or b_t.

If $u \geq s + t - k/2 - (b_s + b_t)$, place retained portions of sizes r_s, r_t on the same side. They can be chosen with union size

$$\max(r_s, r_t, r_s + r_t - u) \leq k/2,$$

by overlapping them as much as their common region permits. Fill this side to size $k/2$ and put both outside types on the opposite side. Again the deleted spoke masses satisfy the desired bounds. The two cases cover all values of u because $2(b_s + b_t) \geq s + t - k/2$.

To round the fractional cut, let $x_v \in [0, 1]$ be the fraction of core vertex v assigned to the first side, so $\sum_v x_v = k/2$. Keep the outside-type orientations fixed. The spoke cost is an affine function $d + \sum_v w_v x_v$; the weights w_v may be negative. Let A contain the $\lfloor k/2 \rfloor$ smallest weights, and let $b \notin A$ have the smallest remaining weight $a = w_b$. For $k = 0$ there is nothing to round. For $k > 0$, the ordering gives

$$\sum_v w_v x_v - \sum_{v \in A} w_v \geq a \left(\sum_v x_v - |A| \right).$$

Indeed, for $v \in A$ we have $w_v(x_v - 1) \geq a(x_v - 1)$, whereas for $v \notin A$ we have $w_v x_v \geq a x_v$.

For even k, the right-hand side is zero, so choose A. For odd k, it is $a/2$: choose A if $a \geq 0$, and choose $A \cup \{b\}$ if $a < 0$. In the latter case the added cost is $a \leq a/2$. In each case the chosen integral part has size $\lfloor k/2 \rfloor$ or $\lceil k/2 \rceil$, and its spoke cost is at most $pb_s + qb_t$.

Delete all within-part edges using the explicit cut cover of Lemma 6. Every such balanced cut has core cost

$$\binom{|A|}{2} + \binom{k - |A|}{2} = \binom{k}{2} - |A|(k - |A|) = c_k.$$

Thus an actual triangle cover has size at most $c_k + pb_s + qb_t$, as required. □

4.2 Deleting the neighborhood shadow

The *neighborhood shadow* is $E(K_S) \cup E(K_T)$. Deleting it meets every triangle centered outside the core. We may then delete further core edges until the residual core is bipartite.

Lemma 8 (An explicit cut outside the neighborhood shadow)**.** *Suppose $s \leq t$ and put $u = |S \cap T|$. For arbitrary multiplicities,*

$$\tau_\triangle(G) \leq \begin{cases} c_k + (s - u)u, & t \leq k/2, \\ \binom{k}{2} - t(k - t) + su - u^2, & t \geq k/2. \end{cases}$$

The two expressions agree when $t = k/2$.

Proof. Write $A = S \setminus T$, $B = T \setminus S$, $U = S \cap T$ and $O = C \setminus (S \cup T)$, with sizes $a = s - u$, $b = t - u$, u, and $o = k - s - t + u$. Delete the entire neighborhood shadow $E(K_S) \cup E(K_T)$. This destroys all outside-centered triangles. In the residual core, keep a cut with A on one side, $B \cup U$ on the other, and z vertices of O on the A side. Its edge count is

$$ab + a(o - z) + tz + z(o - z) = a(k - s) + (k - 2a)z - z^2.$$

Since $a \leq b$, the real maximizer $z_* = k/2 - a$ is nonnegative. Also $a + o = k - t$, so $z_* \leq o$ exactly when $t \leq k/2$. In that case choose an integer z nearest to z_*; the retained edge count is $\lfloor k^2/4 \rfloor - au$. If $t \geq k/2$, choose $z = o$ instead, retaining $ab + to = t(k - t) - su + u^2$ edges. The surviving graph has a bipartite core and no outside-centered triangle. The deleted core edges therefore form a triangle cover. Subtract the retained edge count from $\binom{k}{2}$. These explicit cuts are not asserted to be maximum cuts. □

We also use this construction for other core partitions in the finite certificate. If a core part contains a', b', u', o' vertices from A, B, U, O, set $j = a' + b' + u' + o'$, $\ell_S = a' + u'$ and $\ell_T = b' + u'$. The number of retained edges across the cut and outside the shadow is

$$j(k - j) - \ell_S(s - \ell_S) - \ell_T(t - \ell_T) + u'(u - u').$$

The last term corrects double subtraction of edges in K_U. Consequently deleting all other core edges gives the cover bound

$$\binom{k}{2} - j(k-j) + \ell_S(s-\ell_S) + \ell_T(t-\ell_T) - u'(u-u'). \tag{3}$$

4.3 A cut in the common neighborhood

Lemma 9 (An unbalanced cut inside the common neighborhood)**.** *Put $m = p+q$, $u = |S \cap T|$, and $E = \binom{k}{2}$. For every $0 \le \ell \le u$,*

$$\tau_\triangle(G) \le E + ps + qt - \ell(k+m) + \ell^2 + \tfrac{1}{4}.$$

In particular choose $\ell = \min(u, (k+m)/2)$.

Proof. Put $B = k+m$, $Q(x) = x^2 - Bx$, and choose $j = \min(u, \lfloor B/2 \rfloor)$. Make one core part a j-element subset of $S \cap T$ and place all outside centers on the other side. The explicit cut construction gives a triangle cover F with

$$|F| = \binom{j}{2} + \binom{k-j}{2} + p(s-j) + q(t-j) = E + ps + qt + Q(j).$$

If $u \le \lfloor B/2 \rfloor$, then $j = u$ and, for every $\ell \le u$,

$$Q(\ell) - Q(j) = (u-\ell)(B-u-\ell) \ge 0.$$

Otherwise $j = \lfloor B/2 \rfloor$. Since B is an integer, $0 \le B - 2j \le 1$, and completing the square gives

$$Q(j) = -\frac{B^2}{4} + \frac{(B-2j)^2}{4} \le -\frac{B^2}{4} + \frac{1}{4} \le Q(\ell) + \frac{1}{4}.$$

Thus the same cover F satisfies the asserted bound for all $0 \le \ell \le u$. The minimum of Q on this interval occurs at $\ell = \min(u, B/2)$, yielding in particular

$$\tau_\triangle(G) \le E + ps + qt - \frac{B^2}{4} + \max(0, B/2 - u)^2 + \frac{1}{4}.$$

□

5 The finite comparison

The preceding lemmas give a finite arithmetic test once k, s, t, u, p, q are fixed. Compute the integer lower bound N from (2). A certificate for these parameters chooses one of the following covers:

1. the integer bound $\lfloor c_k + pf_k(s) + qf_k(t) \rfloor$;
2. the shadow cover (3) for a recorded core part;
3. the cut cover in Lemma 6 for a recorded core part.

For the last two choices, the record gives four nonnegative integers a', b', u', o' bounded by $s-u, t-u, u, k-s-t+u$. These inequalities ensure that the part exists in every graph with the specified parameters. The test checks these bounds and that the cover cost is at most $2N$. Every accepted record therefore proves $\tau_\triangle(G) \le 2\nu_\triangle(G)$.

Lemma 10. *Every split graph with at most two active neighborhood types and* $2 \le k \le 42$ *satisfies* $\tau_\triangle(G) \le 2\nu_\triangle(G)$.

Computer-assisted proof. The certificate checks all the integer tuples

$$2 \le k \le 42, \quad 2 \le s \le t \le k, \quad \max(0, s+t-k) \le u \le s,$$
$$1 \le p \le h(s), \qquad 1 \le q \le h(t),$$

together with the one-type cases $2 \le s \le k$, $0 \le p \le h(s)$, represented by $t = u = s$ and $q = 0$. The numbers of checked tuples are

one type or clique	13 622
nested neighborhoods, including identical	3 591 105
equal-size crossing neighborhoods	1 920 614
unequal-size crossing neighborhoods	18 323 030
total	23 848 371

These counts refer to parameter representations, not isomorphism classes. For each tuple the checker first tries the separable cover, then eight shadow cuts: each of A, B, U is placed wholly on either side, and the chosen side is filled with as many vertices of O as possible without exceeding $\lfloor k/2 \rfloor$ when it is not already larger. There are 60 tuples for which these candidates do not suffice; the certificate supplies an explicit cut for each of them. Its validity is checked by the same part-size and cost inequalities.

The resulting range checks are proved in 2326 Lean modules, including their assembly into the quantified statement `TuzaFiniteData.all_checked`. The checker soundness theorem connects acceptance of each record to an actual cover and packing of every graph it represents. The finite sources, range boundaries, and 60 cut records are included in the certificate release [5]. Compression and the parameterization above exhaust the stated graph class, so the checked comparisons prove the lemma. □

6 Analytic reductions for large cliques

We now assume $k \ge 43$ and compressed multiplicities. The first step handles a single type, small neighborhoods, and small total multiplicity.

Lemma 11 (The one-type comparison)**.** *Let* $k \ge 43$, $2 \le d \le k$, *and* $0 \le r \le h(d)$. *Let* H *consist of a* k*-vertex core and* r *outside centers with one common neighborhood of size* d. *Then*

$$2\nu_\triangle(H) \ge c_k + r f_k(d).$$

Proof. The r local colors give $r\lfloor d/2 \rfloor$ outside triangles, including at the endpoint $r = h(d)$. The clique, outside, completion, and sum-coloring bounds therefore give

$$2\nu_\triangle(H) \ge \max\left\{2g(k),\ r(d-1),\ \frac{2c_k + 2r(d-1)}{3},\ \frac{k(k-1)(k-2)/3 + rd(d-1)}{k+r}\right\}.$$

For $d \le k/4$, the clique bound suffices, since $2g(k) - c_k \ge (k^2 - 6k + 5)/12 > 0$. For $k/4 \le d \le 3k/4$ and $r \ge (k+5)/3$, take one quarter of the clique bound and three quarters of the completion bound. Subtracting $c_k + r(d/2 - k/8)$ gives at least

$$\frac{g(k) - c_k}{2} + \frac{r(k-4)}{8} \ge \frac{k-19}{24} > 0.$$

For $d \geq 3k/4$ and $r \geq (k+2)/2$, the outside bound gives

$$r(d-1) - [c_k + r(d-k/2)] = r(k/2-1) - c_k \geq (2k-5)/4 > 0.$$

It remains to justify the sum-coloring comparison in the complementary multiplicity ranges. For a nonnegative real m, define

$$F_d(m) = \frac{k(k-1)(k-2)}{3} + md(d-1) - (k+m)\left(\frac{(k-1)^2}{4} + mf_k(d)\right).$$

The following identities hold in their indicated degree ranges:

$$\begin{aligned}
48F_d(m) &= P(k,m) + 3m(4d-k-m-2)^2, && k/4 \leq d \leq 3k/4,\\
12F_d(m) &= Q(k,m) + 3m(2d-k-m-1)^2, && 3k/4 \leq d \leq k,\\
P(k,m) &= 4k^3 - 24k^2 + 20k - 9mk^2 + 12mk - 12m^2 - 24m - 3m^3,\\
Q(k,m) &= k^3 - 6k^2 + 5k - 3m^3 - 6m^2 - 6m.
\end{aligned}$$

Both P, Q decrease with $m \geq 0$. At their required endpoints,

$$\begin{aligned}
9P(k,(k+5)/3) &= 8h^3 + 690h^2 + 15057h + 6912 > 0, && h = k-43 \geq 0,\\
8Q(k,(k+2)/2) &= 5j^3 + 177j^2 + 1615j + 747 > 0, && j = k-17 \geq 0.
\end{aligned}$$

Thus $F_d(m) \geq 0$ when $k/4 \leq d \leq 3k/4$ and $m \leq (k+5)/3$, or when $d \geq 3k/4$ and $m \leq (k+2)/2$. Taking $m = r$ proves the remaining cases, since $c_k \leq (k-1)^2/4$. The comparison includes $r = 0$ and $r = h(d)$. □

Theorem 12 (Small neighborhoods or small total multiplicity)**.** *Let G have at most two active neighborhood types and a core of size $k \geq 43$. Compress their multiplicities to*

$$p_0 = \min(p, h(s)), \quad q_0 = \min(q, h(t)), \quad m = p_0 + q_0.$$

Then Tuza's inequality holds if either

$$\boxed{\min(s,t) \leq k/4} \qquad \textit{or} \qquad \boxed{3m \leq k+5.}$$

Both statements allow arbitrary overlap; the first places no further restriction on either multiplicity or on the larger neighborhood size.

Proof. Compression preserves both parameters, so work with p_0, q_0. Relabel so that $s \leq t$.

If $s \leq k/4$, then $f_k(s) = 0$, and Lemma 7 gives $\tau_\triangle(G) \leq c_k + q_0 f_k(t)$, independently of p_0. The subgraph on the core and the q_0 centers of type T is the graph of Lemma 11. Its packing is also a packing of G, so $2\nu_\triangle(G) \geq c_k + q_0 f_k(t) \geq \tau_\triangle(G)$.

Now suppose $3m \leq k+5$. The previous case permits us to assume $s, t > k/4$; $m = 0$ is just a clique. Put

$$F_d = \frac{k(k-1)(k-2)}{3} + md(d-1) - (k+m)\left(\frac{(k-1)^2}{4} + mf_k(d)\right).$$

For $k/4 < d \leq 3k/4$, the completed-square identity in the proof of Lemma 11 gives $48F_d = P(k,m) + 3m(4d-k-m-2)^2 \geq 0$, since $m \leq (k+5)/3$ and $k \geq 43$. For $d \geq 3k/4$, the corresponding identity is $12F_d = Q(k,m) + 3m(2d-k-m-1)^2 \geq 0$. Here we use $m \leq (k+5)/3 \leq (k+2)/2$ and the positivity bound for Q proved in the same lemma (already valid for $k \geq 17$).

Average F_s, F_t with weights $p_0/m, q_0/m$. Their weighted sum is

$$\frac{k(k-1)(k-2)}{3} + p_0 s(s-1) + q_0 t(t-1) \\ - (k+m)\left(\frac{(k-1)^2}{4} + p_0 f_k(s) + q_0 f_k(t)\right) \geq 0.$$

The first line divided by $k+m$ is a lower bound for $2\nu_\triangle(G)$ by Lemma 5. The last parenthesis is an upper bound for $\tau_\triangle(G)$ by Lemma 7 and $c_k \leq (k-1)^2/4$. This proves the second assertion. □

Lemma 13. *If $s \leq t \leq k/2$ and $k \geq 43$, then $\tau_\triangle(G) \leq 2\nu_\triangle(G)$.*

Proof. Lemma 8 gives $\tau_\triangle(G) \leq c_k + (s-u)u \leq c_k + k^2/16$. On the other hand,

$$2g(k) - c_k - k^2/16 \geq \frac{k^2 - 24k + 20}{48} > 0 \qquad (k \geq 43).$$

A packing within the core therefore suffices. □

7 A continuous comparison for all remaining sizes

The remaining graphs have two active types. Normalize their parameters by

$$x = s/k, \quad y = t/k, \quad w = u/k, \quad \alpha = p/k, \quad \beta = q/k, \quad \rho = 1/k, \quad \mu = \alpha + \beta.$$

The previous section leaves only points in the domain

$$1/4 \leq x \leq y \leq 1, \qquad y \geq 1/2, \qquad \max(0, x+y-1) \leq w \leq x, \\ 0 \leq \rho \leq 1/43, \qquad \rho \leq \alpha \leq x, \qquad \rho \leq \beta \leq y, \\ 3(\alpha+\beta) \geq 1 + 5\rho. \tag{4}$$

Including boundary points and $\rho = 0$ only enlarges the domain. In particular, proving a statement throughout this domain covers arbitrarily large integer k.

7.1 The bounds to compare

Put $c^+ = (1-\rho)^2/4$ and $f(d) = \max(0, d/2 - 1/8, d - 1/2)$. We express cover upper bounds divided by k^2 and packing lower bounds on $2\nu_\triangle(G)/k^2$. The separable and shadow covers give

$$D_0 = c^+ + \alpha f(x) + \beta f(y), \qquad D_1 = (1-\rho)/2 - y(1-y) + w(x-w).$$

Four more candidates are balanced cuts. Fill a part of normalized size $1/2$ in each of the orders

$$(U, A, B, O), \quad (U, B, A, O), \quad (A, U, O, B), \quad (A, O, U, B).$$

The normalized sizes of these regions are $w, x-w, y-w, 1-x-y+w$. At each step take as much of the next region as the remaining space permits. For the first two orders place both outside types opposite the filled part; for the last two place type S opposite it and type T on its side. The rounding argument in Lemma 7 gives an integral balanced cut of no greater spoke cost. Its normalized cost is bounded by $D_i = c^+ + d_i$, where

$$d_2 = \alpha(x - 1/2)_+ + \beta\big(y - \min(w, 1/2) - (1/2 - x)_+\big)_+, \\ d_3 = \beta(y - 1/2) + \alpha(x - \min(w, 1/2)), \\ d_4 = \alpha(x - 1/2)_+ + \beta \max(y - 1/2, w - (x - 1/2)_+), \\ d_5 = \beta(y - 1/2) + \alpha(w - y + 1/2)_+.$$

These formulas are obtained by counting neighbors placed on each side. For example, in the first order the filled part takes $\min(w, 1/2)$ from U, then fills the rest of S up to a total of $\min(x, 1/2)$; any available space $(1/2 - x)_+$ is filled from B next. The other orders are counted in the same way. The feasibility inequalities give $x - w \le 1/2$ and $1 - y \le 1/2$, which justify taking all of A in the third order and all of $A \cup O$ in the fourth.

Lemma 9 gives one further bound,

$$D_6 = c^+ + \alpha(x - 1/2) + \beta(y - 1/2) - \mu^2/4 + \big((1+\mu)/2 - w\big)_+^2.$$

Set $D = \min_{0 \le i \le 6} D_i$. Then $\tau_\triangle(G)/k^2 \le D$.

For the packing bounds, define

$$\begin{aligned}
I &= \alpha(x-\rho) + \beta(y-\rho) - \frac{\alpha\beta w(w-\rho)}{xy}, \\
J &= \frac{\alpha x(x-\rho) + \beta y(y-\rho) - (\mu - y)_+ w(w-\rho)}{y}, \qquad X = \max(I, J), \\
A(r,z) &= \frac{(1-\rho)(1-2\rho)/3 + rx(x-\rho) + zy(y-\rho)}{1+r+z}, \\
P &= \max\left\{ \frac{(1-\rho)(1-2\rho)}{3},\ X,\ \frac{1-2\rho}{6} + \frac{2X}{3}, A(\alpha,\beta), A(\alpha,0), A(0,\beta) \right\}.
\end{aligned}$$

These satisfy $2\nu_\triangle(G)/k^2 \ge P$. Here I and J are relaxed versions of the independent and coupled allocation bounds. The independent expression $r_S e_S + r_T e_T - r_S r_T e_U$ increases with each rate on $[0,1]^2$, so replacing $p/h(s), q/h(t)$ by $p/s, q/t$ is valid. For the coupled expression, with $s \le t$, increasing the palette size from $h(t)$ to t decreases

$$\frac{p\binom{s}{2} + q\binom{t}{2} - (p+q-H)_+\binom{u}{2}}{H}.$$

Indeed, below $H = p + q$ this equals $\binom{u}{2} + [p(\binom{s}{2} - \binom{u}{2}) + q(\binom{t}{2} - \binom{u}{2})]/H$, and above it the numerator is constant. The remaining terms follow from greedy completion and sum-coloring, discarding integer rounding.

7.2 An auxiliary comparison

Corollary 14 (A common-neighborhood criterion)**.** *Suppose $k \ge 34$ and $p + q = m \le 2k$. Then Tuza's inequality holds if*

$$\boxed{4(k+m)(k+m-2u)_+^2 \le k(k^2 + 3m^2).}$$

After multiplicity compression, the condition $m \le 2k$ is automatic.

Proof. Use $x = s/k$, $y = t/k$, $\alpha = p/k$, $\beta = q/k$, $\mu = m/k$, $\rho = 1/k$, and put $e = ((1+\mu)/2 - u/k)_+$. The preceding cover bound, divided by k^2, is

$$C = \frac{(1-\rho)^2}{4} + \alpha x + \beta y - \frac{\mu}{2} - \frac{\mu^2}{4} + e^2.$$

Let A be the normalized sum-coloring lower bound on $2\nu_\triangle(G)$. Completing squares gives the identity

$$\begin{aligned}
(1+\mu)(A - C) &= B(\mu,\rho) - (1+\mu)e^2 \\
&\quad + \alpha\left(x - \frac{1+\mu+\rho}{2}\right)^2 + \beta\left(y - \frac{1+\mu+\rho}{2}\right)^2, \\
B(\mu,\rho) &= \frac{1}{12} + \frac{\mu^2}{4} - \frac{(1+\mu^2)\rho}{2} + \left(\frac{5}{12} - \frac{\mu}{2}\right)\rho^2.
\end{aligned}$$

For $0 \le \mu \le 2$ and $0 \le \rho \le 1/34$,

$$B(\mu,\rho) \ge \frac{1}{12} - \frac{1}{68} - \frac{7}{12 \cdot 34^2} + \left(\frac{1}{4} - \frac{1}{68}\right)\mu^2 = \frac{315}{4624} + \frac{4}{17}\mu^2 \ge \frac{1+3\mu^2}{16}.$$

The stated criterion is exactly $16(1+\mu)e^2 \le 1+3\mu^2$. Thus $A \ge C$. □

7.3 The interval certificate

Lemma 15. *On the domain* (4)*, $P \ge D$.*

Computer-assisted proof. The certificate partitions a box enclosing (4) into finitely many boxes with rational endpoints. A box may first be contracted using the linear inequalities in (4); each contraction preserves every feasible point. An empty box needs no further check. A nonempty terminal box is accepted by one of two tests.

The first test verifies

$$(1+\mu)\big((1+\mu)/2 - w\big)_+^2 \le (1+3\mu^2)/16$$

throughout the box. The completed-square calculation in Corollary 14 gives $A(\alpha,\beta) \ge D_6$ there: the domain has $\rho \le 1/43 < 1/34$ and $\mu \le 2$. The second test directly compares the other bounds. Subtract c^+ from every packing term and every cover term, preserving their difference. Evaluate each expression by rational interval arithmetic. Accept if the lower endpoint of some packing interval is at least the upper endpoint of some cover interval. The auxiliary identity

$$0 \le w(x-w) \le x^2/4$$

can sharpen the shadow-cover interval. For reference, the clique gap is

$$\frac{(1-\rho)(1-2\rho)}{3} - c^+ = \frac{1}{12} - \frac{\rho}{2} + \frac{5\rho^2}{12}.$$

Addition, multiplication, squaring, minimum, maximum, and division by a positive interval use bounds proved to enclose every possible value. All arithmetic is rational; no floating-point estimate is a proof step. A nonterminal node records a coordinate split and both children. The split-coverage proof combines the two child conclusions. Four adjacent initial slabs cover the domain:

x interval	accepted leaves
$[1/4, 7/16]$	5 604
$[7/16, 5/8]$	116 226
$[5/8, 13/16]$	55 861
$[13/16, 1]$	815
total	178 506

The complete trees, including all 357 008 nodes, are supplied in the certificate sources [5]. Lean checks the terminal comparisons, the soundness of contractions and interval evaluation, and the coverage of every split and root. The resulting theorem `TuzaIntervalData.all_checked` proves the comparison on the whole domain. Every leaf is accepted; no sampled-point inference is used. □

Proof of Theorem 1. Delete inactive vertices and compress multiplicities. Both operations preserve $\tau_\triangle$ and $\nu_\triangle$. For $k \le 1$ the graph has no triangles; for $2 \le k \le 42$ apply Lemma 10.

Suppose $k \ge 43$. The one-type comparison handles zero or one active type. With two types, order $s \le t$. Theorem 12 handles $s \le k/4$ and $3(p+q) \le k+5$; Lemma 13 handles

$t \le k/2$. Every remaining instance normalizes to a point in (4). Lemma 15 and the graph bounds then give

$$\frac{2\nu_{\triangle}(G)}{k^2} \ge P \ge D \ge \frac{\tau_{\triangle}(G)}{k^2}.$$

These cases exhaust the graph class in the theorem. □

8 Formal verification and reproducibility

The public theorem is `TuzaTwoTypes.tuza_of_two_active_types` in `formal/Result.lean`. Its hypotheses state that G is a finite simple graph, C is a clique, its complement is independent, and the active outside neighborhoods form a set of cardinality at most two. The conclusion uses the actual optimum cover and packing numbers.

The formal proof is divided into four main modules:

`GraphReductions`	Graph constructions, multiplicity compression, integer bounds and finite-checker soundness.
`CertificateKernel`	Rational intervals, domain contractions, expression evaluation and tree-checker soundness.
`NormalizedBounds`	Analytic reductions and the connection between normalized inequalities and graph optima.
`Result`	The final theorem, combining both certificate families.

The finite certificate adds 2326 modules. The continuous certificate adds 860: 347 subtree modules, four roots, 508 helper modules, and one domain-coverage module. Together with the four main modules these give 3190 project modules.

The public source build completed on 8 October 2026 with Lean 4.34.0 and Mathlib revision `5ed2965256430c3649e86755f9576b54eca72435`. It compiled every project module from source, with no precompiled project modules as inputs. The run, source hashes, compiler output, and verification report are linked from the repository [5]. The certificate release contains the complete generated Lean sources; the README gives the reproduction commands.

Certificate generators supply proof terms, not axioms. Lean checks the arithmetic proofs with `decide +kernel` and checks the soundness theorems connecting them to the graph statement. The final theorem depends only on `propext`, `Classical.choice`, and `Quot.sound`. The project sources contain no admitted proof, custom axiom, unsafe declaration, or native decision shortcut. A deliberately false arithmetic proof was rejected in the same build. For Mantel's bound the formalization uses Mathlib's Turán theorem; Lemma 4 also gives its elementary degree-sum proof.

Use of AI. The proof development, certificate code, Lean formalization, and manuscript were developed with OpenAI Codex (GPT-6 Astra). The work is published under the pseudonym MathIsEvenEasier. The public build establishes formal verification of the stated Lean theorem; no independent mathematical peer review is claimed.

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